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TRIANGULAR WINGS CAMBERED AND TWISTED TO SUPPORT SPECIFIED DISTRIBUTIONS OF LIFT AT SUPERSONIC SPEEDS

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SUMMARY

The linearized supersonic flow theory has been applied to the calculation of the camber and twist required for various lift distributions on a triangular wing with leading edges behind the Mach cone.

Several specific examples have been derived and plots of the cambered surfaces, the corresponding lift distributions, and values of the theoretical drag coefficients are presented. The examples were chosen to avoid the singularities in lift distribution or camber that appear in previously given solutions and may be useful for practical applications since they avoid large pressure peaks and extreme angles of twist.

INTRODUCTION

The design of cambered wings requires a knowledge of the relation between the shape of the lifting surface and the distribution of lift over it. For airfoils of triangular plan form at supersonic speed, two examples of this relation are known; namely, the lift distribution for a flat wing at an angle of attack (reference 1) and the camber required to support a uniform distribution of lift over the surface (reference 2).

The solution for the flat wing shows an infinite concentration of lift along the leading edges, while that for the uniform lift distribution shows that an extremely large angle of attack at the center line of the triangle would be required. Thus, both solutions show singularities which the designer may wish to avoid. It was suggested to the present author by R. T. Jones that these undesirable singularities in the lift and camber distribution could be avoided, at a particular lift coefficient, by an appropriate choice of the functions used to represent the lift distributions. In the present report, several examples of such nonsingular lift distributions and their corresponding camber shapes have been derived.

Following a brief review of the method, a series of functions capable of representing a variety of lift distributions is presented. It is shown that the desired property of the lift distribution and camber shape can be achieved by an appropriate relation between the coefficients of this series. Numerical examples showing lift distributions and camber shapes which involve the first few terms of the general series are then given.

METHOD

Conical Flow Theory

As in the usual linearized theory (references 3 and 4), the boundary conditions are satisfied in the horizontal plane rather than in the actual plane of the wing. The coordinate system, as depicted in figure 1, is chosen so that the apex of the triangle is at the origin. The positive x axis extends back in the plane of symmetry of the triangle, the z direction is vertical, and the plane of the triangle is approximately horizontal. The symbols used are defined in Appendix A.

The fractional perturbation velocities are denoted by u, v, and w (i.e., fractions of stream velocity) in the x, y, and z directions, respectively. The difference in pressure coefficient between upper and lower surfaces is given by

$$\frac{\Delta p}{q} = 4u$$

and the slope of the surface by

$$\frac{dz}{dx} = w$$

The linearized equation is solved using the method described in reference 4. Solutions are found for a Mach number of $\sqrt{2}$, but may be extended to other Mach numbers by the use of the well-known Prandtl transformation.

In reference 4 it is shown that a special complex variable exists such that if $f(\zeta)$ is any analytic function of ζ, then $u(\zeta)$ = real part of $f(\zeta)$ and $w(\zeta)$ = real part of $\left(-i\int\frac{\sqrt{1-\zeta^2}}{\zeta}\,df\right)$ is a conical solution of the linearized equation.

The complex variable ζ represents a transformation of the conical flow field which causes no distortion of the airfoil coordinates on the $z=0$ plane inside the Mach cone, but places the Mach cone along the real axis in the ζ plane from $\zeta = -\infty$ to -1 and from 1 to ∞. The exact expression for ζ is

$$\zeta = \frac{2\epsilon}{1+\epsilon^2}$$

where

$$\epsilon = \frac{y+iz}{x+\sqrt{x^2-y^2-z^2}}$$

The use of the variable ζ results in a simplification of boundary conditions on the plane $z=0$, since it can be shown that ζ approaches $\frac{y+iz}{x}$ as z approaches zero.

To yield the lifting case, $f(\zeta)$ must be so chosen that its real part will vanish everywhere on the real axis in the ζ plane except on the wing, which extends from $\zeta = -m$ to $\zeta = +m$. In addition, the real part of $f(\zeta)$ must be antisymmetric with respect to the real axis in the ζ plane. If the wing lies entirely within the Mach cone, $f(\zeta)$ must vanish at infinity. This latter requirement arises because in the ζ plane the Mach cone is to be found at infinity in any direction as well as along the real axis as previously described.

For wings inside the Mach cone, the integration for w can be taken along the real axis from $\zeta' = 1$ to $\zeta' = \zeta$ making w zero everywhere on the Mach cone if $f(\zeta)$ is zero at infinity. The term ζ' is a dummy variable of integration replacing ζ in the integrand.

Thus, to represent the lifting surface and its pressure distribution, $f(\zeta)$ must have the following properties:

1. $f(\zeta)$ must be an analytic function of ζ.

2. $f(\zeta)$ must vanish at infinity.

3. The real part of $f(\zeta)$ along the real axis must represent the desired lift distribution of the wing.

4. The real part of $f(\zeta)$ must be antisymmetric with respect to the real axis in the ζ plane.

When checking a prospective $f(\zeta)$ against these four requirements, ζ must be considered a complex variable. However, once these requirements are met, all subsequent calculations are made on the $z=0$ plane, which corresponds to the real axis in the ζ plane, so that ζ is equal to y/x, a real variable. Thus in the final expressions for $u(\zeta)$, $w(\zeta)$, and $\frac{z}{x}(\zeta)$, ζ is equal to y/x.

An example of a function satisfying the foregoing requirement is

$$f(\zeta) = \frac{m}{\sqrt{m^2-\zeta^2}}$$

Here

$$u(\zeta) = \frac{m}{\sqrt{m^2-\zeta^2}} \text{ for } |\zeta| \leqq m$$

and

$$u(\zeta) = 0 \text{ for } |\zeta| \geqq m$$

or

$$u(x,y,0) = \frac{m}{\sqrt{m^2-\frac{y^2}{x^2}}} \text{ over the wing.}$$

Since

$$w(\zeta) = \text{real part } \left(-i \int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\zeta'} df'\right)$$

it is found that

$$w(\zeta) = -\frac{1}{m} E\left(\sqrt{1-m^2}\right) \text{ for } |\zeta| \leqq m$$

and

$$w(\zeta) = \frac{m\sqrt{1-\zeta^2}}{\zeta\sqrt{\zeta^2-m^2}} - \frac{1}{m} E\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right) \text{ for } |\zeta| \geqq m$$

This solution gives the lift distribution for a flat triangle at an angle of attack. (See reference 1.)

As a second example, choose

$$f(\zeta) = i \log \frac{\zeta+m}{\zeta-m}$$

where in this case it is found that

$$u(\zeta) = \pi \text{ for } |\zeta| \leqq m$$

and

$$u(\zeta) = 0 \text{ for } |\zeta| \geqq m$$

Integration shows that

$$w(\zeta) = \frac{\sqrt{1-m^2}}{m}\left(\cosh^{-1}\left|\frac{1+m\zeta}{\zeta+m}\right| + \cosh^{-1}\left|\frac{1-m\zeta}{\zeta-m}\right|\right) - \frac{2}{m}\cosh^{-1}\left|\frac{1}{\zeta}\right|$$

for the whole region. This solution gives the required camber and twist for a uniform lift distribution. (See reference 2.)

General Expressions for Lift, Drag, and Camber

The camber is determined from the relations

$$\frac{dz}{dx} = w(\zeta) = w\left(\frac{y}{x}\right)$$

or

$$z(x,y) = \int_{|y|}^{x} w\left(\frac{y}{x'}\right) dx' \tag{1a}$$

The integration is taken from $x' = |y|$ to $x' = x$ so that the stream surface in which the wing lies may start at $z=0$ on the Mach cone. Thus, the shape of the surface is obtained in the form $z = z(x,y)$.

Since the wing surface is conical, its shape can be expressed in the form $\frac{z}{x} = \frac{z}{x}(\zeta)$, that is, $\frac{z}{x}$ = function of ζ where $\zeta = \frac{y}{x}$. Furthermore, the integration for z can be accomplished in terms of ζ, thus

$$\frac{z}{x}(\zeta) = - |\zeta| \int_1^{|\zeta|} \frac{w(\zeta')}{\zeta'^2} d\zeta' \tag{1b}$$

The magnitude of camber is related to the lift coefficient by finding the common factor of u, w, and $\frac{z}{x}$ which will yield the desired lift coefficient.

$$C_L = \frac{\text{total lift}}{q \times \text{wing area}} = \frac{1}{S} \int_S \int \frac{\Delta p}{q} ds = \frac{4}{S} \int_S \int u \, ds$$

If $u = Bu'$ where B is the desired factor,

$$C_L = \frac{4B}{S} \int_S \int u' \, ds$$

For the symmetrical triangular wing extending from $x=0$ to $x=1$ along the center line and from $y = -m$ to $y = +m$ along the trailing edge, the integration can be accomplished with a single integration in terms of ζ. Since ζ is constant in the element of area enclosed by a triangle the base of which is dy along the trailing edge and the vertex of which is at the vertex of the wing,

$$C_L = \frac{4B}{m} \int_{-m}^{+m} u'(\zeta) \, \frac{1}{2} \, d\zeta$$

and, for a symmetrical distribution,

$$C_L = \frac{4B}{m}\int_0^m u'(\zeta)d\zeta \text{ or } B = \frac{m}{4} \times \frac{C_L}{\int_0^m u'(\zeta)d\zeta} \qquad (2)$$

The coefficient of drag due to lift is obtained as follows:

$$C_D = -\frac{1}{S}\int_S\int \frac{\Delta p}{q} \times \frac{dz}{dx}\, ds = -\frac{4}{S}\int_S\int u \times w ds$$

The integration over the wing can again be reduced to a single integration interms of ζ.

$$C_D = -\frac{4}{m}\int_0^m u(\zeta)w(\zeta)d\zeta \qquad (3)$$

No additional term for leading-edge suction is required because the pressure distributions chosen have no infinite pressure peaks along the leading edges.

SERIES DEVELOPMENT FOR AN ARBITRARY LIFT DISTRIBUTION

Elements Suitable for Series Representation

The analytic functions

$$f_o(\zeta) = i \log \frac{\zeta+m}{\zeta-m} \qquad (4)$$

and

$$f_{2n-1}(\zeta) = i\left(\frac{\zeta-\sqrt{\zeta^2-m^2}}{m}\right)^{2n-1}, \text{ n=1, 2, 3, etc.} \qquad (5)$$

can be superimposed to yield an arbitrary lift distribution extending from $-m$ to $+m$. The function of f_{2n-1} is similar to that given in reference 5. The exponent is restricted to odd integers so that the lift distribution will be an even function of ζ, or symmetric with respect to the center line of the triangle. Such a solution would be represented by a series of the form

$$f(\zeta) = B\, i \left[a_0 \log \frac{\zeta+m}{\zeta-m} + \sum_{n=1}^{\infty} b_{2n-1}\left(\frac{\zeta-\sqrt{\zeta^2-m^2}}{m}\right)^{2n-1}\right]$$

The velocity components corresponding to the individual terms are

$$u_{2n-1}(\zeta) = \text{real part}\left[f_{2n-1}(\zeta)\right] \tag{6}$$

and

$$w_{2n-1}(\zeta) = \text{real part}\left(-i\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\zeta'}\, df'_{2n-1}\right) \tag{7}$$

It can be seen that this family of functions has the required properties of being analytic, vanishing at infinity, and having real parts which are antisymmetric with respect to the real axis in the ζ plane. Also the u_{2n-1} terms are zero everywhere on the real axis except at $|\zeta| \leqq m$.

As an example of the requirement that the real part of $f(\zeta)$ be antisymmetric with respect to the real axis, it will be shown that $i(\zeta-\sqrt{\zeta^2-m^2})$ meets this requirement. For this purpose ζ can be taken equal to $\frac{y+iz}{x}$ since the argument can be restricted to the region where z approaches zero. It is immediately apparent that the real part of $i\zeta$ is antisymmetric with respect to the real axis. That the real part of $i\sqrt{\zeta^2-m^2}$ has this symmetry is not obvious but it can be seen as follows:

Let

$$\zeta+m = r_1 e^{i\varphi_1}, \quad \zeta-m = r_2 e^{i\varphi_2}$$

where r_1 is the length of the line joining the point, $-m$, on the real axis to the point, ζ, in the ζ plane; and φ_1 is the angle this line makes with the positive direction along the real axis. Similarly, r_2 is the length of the line joining $+m$ to ζ and φ_2 is the angle this line makes with the positive real axis.

Then

$$\begin{aligned}
\zeta^2-m^2 &= (\zeta+m)\ (\zeta-m) \\
&= r_1 e^{i\varphi_1}\ r_2 e^{i\varphi_2} \\
&= r_1 r_2 e^{i(\varphi_1+\varphi_2)}
\end{aligned}$$

$$\begin{aligned}
i\sqrt{\zeta^2-m^2} &= i\sqrt{r_1 r_2}\ e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)} \\
&= i\sqrt{r_1 r_2}\left[\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + i\ \sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]
\end{aligned}$$

and

$$\text{real part } \left(i\sqrt{\zeta^2-m^2}\right) = -\sqrt{r_1 r_2}\ \sin\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

From the definitions of φ_1 and φ_2, it follows that the real part of $i\sqrt{\zeta^2-m^2}$ is always negative above the real axis and always positive below. This is seen by starting at a point above the real axis with φ_1 and φ_2 less than π and following a path which does not pass through that part of the real axis between $+m$ and $-m$. Following a path that does pass between $+m$ and $-m$ is prohibited because the real part of $f(\zeta)$ will be discontinuous there if it is antisymmetric. The real part of $f(\zeta)$ is not discontinuous along the real axis outside of $+m$ and $-m$ because it passes through zero there.

Orthogonality Relations

The u_{2n-1} terms are orthogonal in the interval $-m$ to $+m$ and are zero at the limits of the interval. Any reasonable even conical lift distribution can be expanded in an infinite series of u_o and u_{2n-1}.

Let

$$\zeta = m \sin \theta$$

for $|\zeta| \leqq m$ on the real axis, then

$$\begin{aligned} f_{2n-1}(m \sin \theta) &= i(\sin \theta - i \cos \theta)^{2n-1} \\ &= i^{1-2n+1}(\cos \theta + i \sin \theta)^{2n-1} \\ &= (-1)^{n-1} e^{i(2n-1)\theta} \end{aligned}$$

and finally

$$f_{2n-1}(m \sin \theta) = (-1)^{n-1} [\cos(2n-1)\theta + i \sin(2n-1)\theta] \tag{8}$$

since

$$u_{2n-1}(m \sin \theta) = \text{real part } [f_{2n-1}(m \sin \theta)]$$

$$u_{2n-1}(m \sin \theta) = (-1)^{n-1} \cos(2n-1)\theta \tag{9}$$

To establish the orthogonality of these functions, it is necessary to evaluate $\int_0^{\frac{\pi}{2}} u_{2n-1}\, u_{2r-1}\, d\theta$:

$$\int_0^{\frac{\pi}{2}} u_{2n-1}\, u_{2r-1}\, d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2n-1)\theta \cos(2r-1)\theta\, d\theta$$

or

$$\int_0^{\frac{\pi}{2}} u_{2n-1}\, u_{2r-1}\, d\theta = \begin{Bmatrix} 0, & n \neq r \\ \frac{\pi}{4}, & n = r \end{Bmatrix} \tag{10}$$

Equation (10) is the required orthogonality relation.

Let $h(\zeta)$ be any even function which is zero at $\zeta = \pm m$.

Expanding $h(m \sin \theta)$ in the Fourier series,

$$h(m \sin \theta) = \sum_{n=1}^{\infty} b_{2n-1} (-1)^{n-1} \cos(2n-1)\theta$$

and

$$b_{2n-1} = (-1)^{n-1} \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} h(m \sin \theta) \cos(2n-1)\theta \, d\theta \tag{11}$$

The term $a_o u_o = a_o \pi$ can be added to $h(m \sin \theta)$ to give a lift distribution which is not zero at $\zeta = \pm m$.

Then

$$u'(m \sin \theta) = a_o \pi + \sum_{n=1}^{\infty} b_{2n-1} (-1)^{n-1} \cos(2n-1)\theta \tag{12}$$

or

$$u'(\zeta) = a_o \pi + \sum_{n=1}^{\infty} b_{2n-1} u_{2n-1}(\zeta) \tag{13}$$

Since the real part of $\left(i \log \frac{\zeta+m}{\zeta-m}\right) = \pi$ for $|\zeta| \leqq m$ and the terms u_{2n-1} are the real parts of $i\left(\frac{\zeta-\sqrt{\zeta^2-m^2}}{m}\right)^{2n-1}$ it can be seen that the $f'(\zeta)$ which corresponds to $u'(\zeta)$ in equation (13) is

$$f'(\zeta) = i \left[a_o \log \frac{\zeta+m}{\zeta-m} + \sum_{n=1}^{\infty} b_{2n-1} \left(\frac{\zeta-\sqrt{\zeta^2-m^2}}{m}\right)^{2n-1} \right] \tag{14}$$

which in terms of $\zeta = m \sin \theta$ is

$$f'(m \sin \theta) \doteq i\, a_0 \log \frac{\sin \theta + 1}{\sin \theta - 1} + \sum_{n=1}^{\infty} b_{2n-1} (-1)^{n-1} \left[\cos(2n-1)\theta \right.$$

$$\left. + i \sin(2n-1)\theta \right] \tag{15}$$

By the use of the orthogonality relation (equation (10)) it can be shown that no net lift is developed by $u_{2n-1}(\zeta)$ when n is greater than one.

The total lift is proportional to

$$\int_S \int u\,ds = \int_0^m u(\zeta)\,d\zeta$$

then

$$\int_0^m u_{2n-1}(\zeta)\,d\zeta = \int_0^{\frac{\pi}{2}} u_{2n-1}(m \sin \theta)\, m \cos \theta \, d\theta$$

$$= (-1)^{n-1} m \int_0^{\frac{\pi}{2}} \cos(2n-1)\theta \cos \theta \, d\theta$$

$$= m \int_0^{\frac{\pi}{2}} u_{2n-1}\, u_1 \, d\theta$$

and

$$\int_0^m u_{2n-1}(\zeta)\,d\zeta = \begin{Bmatrix} \frac{m\pi}{4}, & n=1 \\ 0, & n>1 \end{Bmatrix} \tag{16}$$

From this relation the factor B, which relates the magnitude of camber to the lift coefficient, can be expressed in terms of a_o and b_1, the first two coefficients of the Fourier expansion of $u'(\zeta)$.

Therefore, if

$$u(\zeta) = B\left[a_o\pi + \sum_{n=1}^{\infty} b_{2n-1}\, u_{2n-1}(\zeta)\right]$$

then

$$B = \frac{m}{4}\,\frac{C_L}{\int_0^m u'(\zeta)d\zeta} = \frac{m}{4}\,\frac{C_L}{\left(a_o m\pi + b_1\frac{m\pi}{4}\right)}$$

or

$$B = \frac{C_L}{\pi(4a_o+b_1)} \tag{17}$$

Condition for a Finite Value of $w(\zeta)$ at $\zeta=0$

The expressions for w_{2n-1} and w_o all have the same type of infinity at $\zeta=0$, namely, $\cosh^{-1}\frac{1}{\zeta}$. By an appropriate relation between the coefficients of the Fourier series, the resultant coefficient of $\cosh^{-1}\frac{1}{\zeta}$ may be made to vanish and the infinity in $w(\zeta)$ at $\zeta=0$ will thereby be avoided.

According to A. Busemann in reference 3, the necessary and sufficient condition that there be no infinity in w at $\zeta=0$ is that $df=0$ at $\zeta=0$.

Differentiating equation (15),

$$(df)_{\theta=0} = i\left[2a_o + \Sigma\,(-1)^{n-1}(2n-1)b_{2n-1}\right]d\theta \quad (\zeta=0 \text{ when } \theta=0)$$

Therefore the necessary and sufficient condition that there be no infinity in w at $\zeta=0$ is

$$2a_o + \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)b_{2n-1} = 0 \tag{18}$$

This relation restricts the lift distribution to the extent that the value of u at $\zeta=0$ must be less than the average value, provided that no negative lift is allowed at any part of the wing. Also the value of u at $\zeta=m$ depends on the shape of the distribution of u over the whole wing.

Examples of Functions Contained in the Series Development

Evaluation of $w(\zeta)$ and $\frac{z}{x}(\zeta)$ becomes increasingly complicated for higher values of n. The following is a list of the expressions for $u(\zeta)$, $w(\zeta)$, and $\frac{z}{x}(\zeta)$ of the first three terms of the series: (They were derived by the use of equations (1a), (1b), (4), (5), (6), and (7). A sample derivation is given in Appendix B.)

$$u_o(\zeta) = \pi$$

$$w_o(\zeta) = \frac{\sqrt{1-m^2}}{m}\left(\cosh^{-1}\left|\frac{1+m\zeta}{\zeta+m}\right| + \cosh^{-1}\left|\frac{1-m\zeta}{\zeta-m}\right|\right) - \frac{2}{m}\cosh^{-1}\left|\frac{1}{\zeta}\right|$$

$$\frac{z_o}{x}(\zeta) = \frac{\sqrt{1-m^2}}{m}\left(\frac{\zeta+m}{m}\cosh^{-1}\left|\frac{1+m\zeta}{\zeta+m}\right| - \frac{\zeta-m}{m}\cosh^{-1}\left|\frac{1-m\zeta}{\zeta-m}\right|\right) - \frac{2}{m}(\infty)$$

$$u_1(\zeta) = \frac{\sqrt{m^2-\zeta^2}}{m}$$

$$w_1(\zeta) = \frac{1}{m}\sqrt{1-\zeta^2} - \frac{1}{m}\cosh^{-1}\left|\frac{1}{\zeta}\right| + \frac{1}{m}\left[F\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)\right.$$

$$\left. - E\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)\right]$$

$$\frac{z_1}{x}(\zeta) = \frac{1}{m}\left(\sqrt{1-\zeta^2} - |\zeta| \cos^{-1}\zeta\right) + \frac{1}{m}\left[K(\sqrt{1-m^2}) - E(\sqrt{1-m^2})\right] - \frac{1}{m}(\infty)$$

$$u_3(\zeta) = \frac{4\zeta^2-m^2}{m^3}\sqrt{m^2-\zeta^2}$$

$$w_3(\zeta) = -\frac{3}{m}\sqrt{1-\zeta^2} - \frac{4}{m^3}(1-\zeta^2)^{3/2} + \frac{4+m^2}{m^3} E\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)$$

$$-\frac{5}{m} F\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right) - \frac{4}{m^3}\zeta\sqrt{1-\zeta^2}\sqrt{\zeta^2-m^2} + \frac{3}{m}\cosh^{-1}\left|\frac{1}{\zeta}\right|$$

$$\frac{z_3}{x}(\zeta) = -\frac{3}{m}\left(\sqrt{1-\zeta^2} - |\zeta| \cos^{-1}\zeta\right) - \frac{2}{m^3}\left[(2+\zeta^2)\sqrt{1-\zeta^2} - 3|\zeta|\cos^{-1}\zeta\right]$$

$$+\frac{4+m^2}{m^3} E(\sqrt{1-m^2}) - \frac{5}{m} K(\sqrt{1-m^2}) + \frac{3}{m}(\infty)$$

The preceding expression for $\frac{z}{x}(\zeta)$ is derived from an integration of only those terms of w which are variable over the range $|\zeta| \leqq m$ plus a constant term which is needed along the line, $\zeta=0$.

For each $\frac{z}{x}$, there will be one term from the integration of $\cosh^{-1}\frac{1}{\zeta}$, which is infinite at $\zeta=0$. When the individual solutions are superimposed in a series, the $\cosh^{-1}\frac{1}{\zeta}$ terms are removed by imposing equation (18). Terms of $\frac{z}{x}(\zeta)$ which are to be removed by equation (18) have been denoted by the symbol (∞) in the table.

The elliptic integrals F and E are complete and hence constant for $|\zeta| \leqq m$. Also $\sqrt{\zeta^2-m^2}$ is taken to be zero for $|\zeta| \leqq m$, since w was to be only the real part of the function resulting from an integration.

An arbitrary dihedral angle can be added to the wing surface along the wing center line without changing the lift distribution as long as no part of the wing is far from the $z=0$ plane. Thus a term, a constant times ζ, can be added to the equation of the surface without changing $w(\zeta)$ or $u(\zeta)$. The value of this constant determines the dihedral angle along the wing center line. It can be seen mathematically that a term $\frac{z}{x}$ equals constant times ζ will be proportional to $\frac{y}{x}$, and w (which equals $\frac{dz}{dx}$) will be zero. Such a term added to the equation of the wing surface has the effect of shifting the stream sheet in which the wing lies so that it does not pass through $z=0$ on the Mach cone. However, such a departure is allowable under the condition that no part of the wing surface sheet lies far from the $z=0$ plane. This represents an approximation which is valid to the same order of magnitude as the practice of calculating $\frac{dz}{dx}$ in the $z=0$ plane rather than in the wing surface. For example, Stewart's solution for the flat triangle yields a V-shaped wing-surface cross section which is equivalent to a flat triangle when the dihedral angle is removed.

Examples of Specific Lift Distributions

Four cases have been worked out using $f_0(\zeta)$, $f_1(\zeta)$, and $f_3(\zeta)$ in various combinations which comply with equation (18). The expressions for $f(\zeta)$, $u(\zeta)$, $w(\zeta)$, and $\frac{z}{x}(\zeta)$ are listed in each case. Graphs are drawn of u and $\frac{z}{x}$ for $C_L=1.0$, $m=0.577$ in figure 2. For purposes of comparison, the flat triangle and constant load triangle are also included in the figure.

The wing-surface shapes of figure 2 are drawn to a square scale. (The unit of length in the horizontal direction is equal to the vertical unit of length.) The $z=0$ plane is the horizontal line above the wing. The semispan of the wing is in each case equal to 0.577 of the radius of the Mach circle. The leading point of the wing would be at the origin of the coordinate system, a distance ahead of the plane depicted in the figure equal to the radius of the Mach circle in that plane.

The cases considered are most easily evaluated by the use of the following tabulated expressions for u, w, and $\frac{z}{x}$:

$$u(\zeta) = B\left[a_0u_0 + \sum_{n=1}^{\infty} b_{2n-1}\, u_{2n-1}\,(\zeta)\right]$$

$$w(\zeta) = B \left[a_0 w_0 + \sum_{n=1}^{\infty} b_{2n-1}\, w_{2n-1}\,(\zeta) \right]$$

$$\frac{z}{x}(\zeta) = B \left[a_0 \frac{z_0}{x} + \sum_{n=1}^{\infty} b_{2n-1}\, \frac{z_{2n-1}}{x}\,(\zeta) \right]$$

The term b_{2n-1} will be zero for $n>2$ in the cases considered here.

The coefficient a_o may be taken equal to 1 or 0. If a_o is equal to 0, b_1 may be taken as 1, and b_3 can be chosen with a value such that the $\cosh^{-1}\frac{1}{\zeta}$ term does not appear in $w'(\zeta)$ or the (∞) term does not appear in $\frac{z'}{x}(\zeta)$. This choice is equivalent to satisfying equation (18).

From equation (17),

$$B = \frac{C_L}{\pi(4a_o+b_1)}$$

The coefficient of drag due to lift was found by a graphical integration of equation (3).

A term $D|\zeta|$ is added to $\frac{z}{x}(\zeta)$ to make allowance for the arbitrary dihedral angle at the center line of the triangle. For example, D can be given the value required to place the leading edges at the same vertical height as the center line. This value was used in the cases considered here.

Case I

$$f(\zeta) = \frac{C_L}{2\pi} i \left[\log \frac{\zeta+m}{\zeta-m} - \frac{2}{m} \left(\zeta - \sqrt{\zeta^2 - m^2}\right) \right]$$

$$u(\zeta) = \frac{C_L}{2\pi} \left(\pi - \frac{2}{m} \sqrt{m^2 - \zeta^2} \right)$$

$$w(\zeta) = \frac{C_L}{2\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left(\cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| + \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right) - \frac{2}{m} \sqrt{1-\zeta^2} \right.$$

$$\left. - \frac{2}{m} \left[K\left(\sqrt{1-m^2}\right) - E\left(\sqrt{1-m^2}\right) \right] \right\}$$

$$\frac{z}{x}(\zeta) = \frac{C_L}{2\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left(\frac{\zeta+m}{m} \cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| - \frac{\zeta-m}{m} \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right) \right.$$

$$\left. - \frac{2}{m} \left(\sqrt{1-\zeta^2} - |\zeta| \cos^{-1}\zeta \right) - \frac{2}{m} \left[K\left(\sqrt{1-m^2}\right) - E\left(\sqrt{1-m^2}\right) \right] - D|\zeta| \right\}$$

$C_L = 1$ $m = 0.577$ $D = 1.486$ $C_D = 0.25$

ζ	$\frac{z}{x}$
0	−0.459
.05	−.429
.10	−.404
.15	−.383
.20	−.366
.22	−.361
.24	−.356
.26	−.353
.28	−.350
.30	−.347
.35	−.346
.40	−.350
.45	−.362
.50	−.384
.55	−.421
.577	−.459

ζ	u
0	0.182
.1	.188
.2	.201
.3	.228
.4	.271
.5	.341
.577	.500

Case II

$$f(\zeta) = \frac{C_L}{4\pi} i \left[\log \frac{\zeta+m}{\zeta-m} + \frac{2}{3m^3} \left(\zeta - \sqrt{\zeta^2-m^2}\right)^3 \right]$$

$$u(\zeta) = \frac{C_L}{4\pi} \left[\pi + \frac{2}{3m^3} (4\zeta^2-m^2) \sqrt{m^2-\zeta^2} \right]$$

$$w(\zeta) = \frac{C_L}{4\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left[\cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| + \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right] - \frac{2}{m} \sqrt{1-\zeta^2} \right.$$

$$\left. - \frac{8}{3m^3} (1-\zeta^2)^{3/2} + \frac{2}{3m^2} \left[(4 + m^2)\, E\left(\sqrt{1-m^2}\right) - 5m^2\, K\left(\sqrt{1-m^2}\right) \right] \right\}$$

$$\frac{z}{x}(\zeta) = \frac{C_L}{4\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left(\frac{\zeta+m}{m} \cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| - \frac{\zeta-m}{m} \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right) \right.$$

$$- \frac{2}{m} \left[\sqrt{1-\zeta^2} - |\zeta| \cos^{-1} \zeta \right] - \frac{4}{3m^3} \left[(2+\zeta^2) \sqrt{1-\zeta^2} - 3 |\zeta| \cos^{-1} \zeta \right]$$

$$\left. + \frac{2}{3m^3} \left[(4+m^2) E\left(\sqrt{1-m^2}\right) - 5m^2 K\left(\sqrt{1-m^2}\right) \right] - D |\zeta| \right\}$$

$C_L = 1$ $\quad$ $m = 0.577$ $\quad$ $D = 22.481$ $\quad$ $C_D = 0.26$

ζ	$\frac{z}{x}$
0	−0.546
.05	−.489
.10	−.442
.15	−.406
.20	−.380
.22	−.373
.24	−.367
.26	−.363
.28	−.361
.30	−.360
.35	−.365
.40	−.382
.45	−.410
.50	−.449
.55	−.504
.577	−.546

ζ	u
0	0.196
.1	.204
.2	.224
.3	.253
.4	.286
.5	.302
.577	.250

Case III

$$f(\zeta) = \frac{C_L}{\pi}\, i\left[\frac{1}{m}\left(\zeta-\sqrt{\zeta^2-m^2}\right) + \frac{1}{3m^3}\left(\zeta-\sqrt{\zeta^2-m^2}\right)^3\right]$$

$$u(\zeta) = \frac{C_L}{\pi}\,\frac{2}{3m^3}\left(2\zeta^2+m^2\right)\sqrt{m^2-\zeta^2}$$

$$w(\zeta) = \frac{C_L}{\pi}\left\{-\frac{4}{3m^3}\left(1-\zeta^2\right)^{3/2} + \frac{2}{3m^3}\left[\left(2-m^2\right) E\left(\sqrt{1-m^2}\right) - m^2 K\left(\sqrt{1-m^2}\right)\right]\right\}$$

$$\frac{z}{x}(\zeta) = \frac{C_L}{\pi}\left\{-\frac{2}{3m^3}\left[\left(2+\zeta^2\right)\sqrt{1-\zeta^2} - 3|\zeta|\cos^{-1}\zeta\right]\right.$$

$$\left. + \frac{2}{3m^3}\left[\left(2-m^2\right) E\left(\sqrt{1-m^2}\right) - m^2 K\left(\sqrt{1-m^2}\right)\right] - D|\zeta|\right\}$$

$C_L = 1$ $\quad m = 0.577$ $\quad D = 10.50$ $\quad C_D = 0.28$

ζ	$\frac{z}{x}$
0	−0.633
.05	−.549
.10	−.481
.15	−.429
.20	−.394
.22	−.385
.24	−.378
.26	−.373
.28	−.372
.30	−.372
.35	−.385
.40	−.414
.45	−.457
.50	−.515
.55	−.588
.577	−.633

ζ	u
0	0.212
.1	.224
.2	.242
.3	.279
.4	.300
.5	.265
.577	0

Case IV

$$f(\zeta) = \frac{C_L}{3\pi} i \left[\log \frac{\zeta+m}{\zeta-m} - \frac{1}{m} (\zeta - \sqrt{\zeta^2-m^2}) + \frac{1}{3m^3} (\zeta - \sqrt{\zeta^2-m^2})^3 \right]$$

$$u(\zeta) = \frac{C_L}{3\pi} \left[\pi - \frac{4}{3m^3} (m^2-\zeta^2)^{3/2} \right]$$

$$w(\zeta) = \frac{C_L}{3\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left[\cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| + \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right] \right.$$

$$- \frac{2}{m} \sqrt{1-\zeta^2} - \frac{4}{3m^3} (1-\zeta^2)^{3/2}$$

$$\left. + \frac{4}{3m^3} \left[(1+m^2) E (\sqrt{1-m^2}) - 2m^2 K (\sqrt{1-m^2}) \right] \right\}$$

$$\frac{z}{x} (\zeta) = \frac{C_L}{3\pi} \left\{ \frac{\sqrt{1-m^2}}{m} \left(\frac{\zeta+m}{m} \cosh^{-1} \left| \frac{1+m\zeta}{\zeta+m} \right| - \frac{\zeta-m}{m} \cosh^{-1} \left| \frac{1-m\zeta}{\zeta-m} \right| \right) \right.$$

$$- \frac{2}{m} (\sqrt{1-\zeta^2} - |\zeta| \cos^{-1} \zeta) - \frac{2}{3m^3} [(2+\zeta^2)\sqrt{1-\zeta^2} - 3|\zeta| \cos^{-1}\zeta]$$

$$\left. + \frac{4}{3m^3} [(1+m^2) E (\sqrt{1-m^2}) - 2m^2 K (\sqrt{1-m^2})] - D |\zeta| \right\}$$

$C_L = 1$ $m = 0.577$ $D = 11.984$ $C_D = 0.26$

ζ	$\frac{z}{x}$	ζ	$\frac{z}{x}$	ζ	u
0	−0.517	0.28	−0.357	0	0.192
.05	−.469	.30	−.356	.1	.198
.10	−.430	.35	−.359	.2	.216
.15	−.398	.40	−.371	.3	.245
.20	−.376	.45	−.394	.4	.281
.22	−.369	.50	−.428	.5	.316
.24	−.364	.55	−.476	.577	.333
.26	−.360	.577	−.517		

EVALUATION OF RESULTS

As can be seen in figure 2, the theoretical coefficients of drag due to lift of the camber shapes considered here are all larger than that of the flat triangle. In practice, however, it is possible that case II and case IV would have lower drag than the flat triangle at some values of lift coefficient because of the suppression of large adverse pressure gradients.

Quite possibly case II and case IV would be capable of attaining a higher lift coefficient than the flat triangle because of their favorable pressure gradients, although the flat triangle pressure distribution would be superimposed on their pressures at lift coefficients other than that for which they are designed.

Case III has the most radical camber, which might cause uncalculated additions to the drag as well as flow separation.

Theoretically the dihedral angle in the surface shape is arbitrary. In practice, however, variation of this parameter might result in an optimum of characteristics at other angles than those chosen in figure 2.

While the linear theory cannot be expected to apply for values of lift coefficient greater than about 0.25, the calculations were made at a lift coefficient of one to better illustrate the wing-surface shapes. This value of lift coefficient, 0.25, is also an approximate limit beyond which the assumption that no part of the wing lies far from the $z = 0$ plane is invalid.

These solutions are applicable to swept-back or other wings obtained by cutting out the rear of the triangle along lines such that no part of the remaining wing lies within the zone of influence of the parts removed.

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APPENDIX A

Symbols

B common factor of u, w, and z which relates the magnitude of camber to the lift coefficient

C_D	coefficient of drag due to lift
C_L	lift coefficient
D	constant, the value of which determines the arbitrary dihedral angle at the center line of the triangle
E	elliptic integral of the second kind (either complete of incomplete)
ϵ	complex variable equal to $\dfrac{y+iz}{x+\sqrt{x^2-y^2-z^2}}$
$f(\zeta)$	analytic function of the complex variable ζ such that the real part of $f(\zeta)$ equals u
F	incomplete elliptic integral of the first kind
K	complete elliptic integral of the first kind
m	horizontal slope of the leading edge of the wing
M	Mach number (equal to $\sqrt{2}$)
$\dfrac{\Delta p}{q}$	the difference in pressure coefficient between upper and lower wing surface
q	dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$
S	wing area
u,v,w	fractional perturbation velocities as fractions of the stream velocity
u'	u divided by B
V	stream velocity
x,y,z	Cartesian coordinates
ζ	equal to $\dfrac{y}{z}$, also used as complex variable equal to $\dfrac{2\epsilon}{1+\epsilon^2}$
ζ', x'	dummy variables used in integrations when a limit of integration is ζ or x

| | absolute value

(∞) terms in the equation of the wing surface which are to be removed by the superposition of other independent solutions

APPENDIX B

SAMPLE CALCULATION OF EQUATIONS IN TABLE OF u, w, AND $\frac{z}{x}$

From equation (5)

$$f_1(\zeta) = i\left(\frac{\zeta - \sqrt{\zeta^2 - m^2}}{m}\right)$$

Since the subsequent calculations are made in the plane $z = 0$, ζ is a real variable equal to $\frac{y}{x}$.

$$df_1 = i\left(\frac{1}{m} - \frac{\zeta}{m\sqrt{\zeta^2 - m^2}}\right)d\zeta$$

From equation (7)

$$w_1 = \text{real part}\left[-i\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\zeta'}\, df'_1\right]$$

$$w_1 = \frac{1}{m}\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\zeta'}\, d\zeta' - \frac{1}{m}\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\sqrt{\zeta'^2 - m^2}}\, d\zeta' \tag{B1}$$

$$\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\zeta'}\, d\zeta' = \sqrt{1-\zeta^2} - \cosh^{-1}\left|\frac{1}{\zeta}\right| \tag{B2}$$

$$\int_1^{\zeta} \frac{\sqrt{1-\zeta^2}}{\sqrt{\zeta'^2 - m^2}}\, d\zeta' = \int_1^{m} \frac{\sqrt{1-\zeta^2}}{\sqrt{\zeta^2 - m^2}}\, d\zeta + \int_m^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\sqrt{\zeta'^2 - m^2}}\, d\zeta'$$

$$\int_1^{\zeta} \frac{\sqrt{1-\zeta'^2}}{\sqrt{\zeta'^2 - m^2}}\, d\zeta' = \int_1^{m} \frac{\sqrt{1-\zeta^2}}{\sqrt{\zeta^2 - m^2}}\, d\zeta \quad \text{for } \zeta \leqq m \tag{B3}$$

The last equality holds because the second term on the right of equation (B3) is imaginary for $\zeta \leqq m$

With the substitution, $r^2 = \frac{1-\zeta^2}{1-m^2}$

$$\int_1^m \frac{\sqrt{1-\zeta^2}}{\sqrt{\zeta^2-m^2}}\,d\zeta = -(1-m^2)\int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}\,\sqrt{1-(1-m^2)r^2}}$$

$$=\int_0^1 \frac{\sqrt{1-(1-m^2)r^2}\,dr}{\sqrt{1-r^2}} - \int_0^1 \frac{dr}{\sqrt{1-r^2}\,\sqrt{1-(1-m^2)r^2}}$$

The two terms on the right of the last equality are recognized as complete elliptic integrals of the second and first kind, respectively, so that

$$\int_1^\zeta \frac{\sqrt{1-\zeta'^2}}{\sqrt{\zeta'^2-m^2}}\,d\zeta' = E(\sqrt{1-m^2}) - K(\sqrt{1-m^2}) \quad \text{for} \quad \zeta \leqq m \tag{B4}$$

The same substitution leads to the result

$$\int_1^\zeta \frac{\sqrt{1-\zeta'^2}}{\sqrt{\zeta'^2-m^2}}\,d\zeta' = E\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)$$

$$- F\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right) \text{for} \quad \zeta \geqq m \tag{B5}$$

These expressions all apply for ζ negative as well as positive because w_1 is an even function about $\zeta = 0$.

Thus from (B1), (B2), (B3), (B4), and (B5)

$$w_1 = \frac{1}{m}\left\{\sqrt{1-\zeta^2} - \cosh^{-1}\left|\frac{1}{\zeta}\right| - E(\sqrt{1-m^2}) + K(\sqrt{1-m^2})\right\} \text{for } |\zeta| \leqq m$$

and

$$w_1 = \frac{1}{m}\left\{\sqrt{1-\zeta^2} - \cosh^{-1}\left|\frac{1}{\zeta}\right| - E\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)\right.$$

$$\left. + F\left(\sin^{-1}\frac{\sqrt{1-\zeta^2}}{\sqrt{1-m^2}}, \sqrt{1-m^2}\right)\right\} \quad \text{for} \quad |\zeta| \geqq m$$

From equation (1b)

$$\frac{z_1}{x} = -|\zeta|\int_1^{|\zeta|} \frac{w_1(\zeta')}{\zeta^2}\, d\zeta' \tag{B6}$$

Since the dihedral angle at the center of the triangle is arbitrary, the elliptic integrals, which are complete and hence constant for $|\zeta| \leqq m$, need not be included in the integration. However, to achieve the correct angle of attack at the center line, they must be taken into account. This operation can be accomplished most simply by the use of equation (1a), which is

$$z = \int_{|y|}^{x} w\left(\frac{y}{x'}\right)\, dx'$$

Thus along the center line, the term of z_1 from the elliptic integrals in the expression for w_1 is

$$\int_0^x \frac{1}{m}\left[K(\sqrt{1-m^2}) - E(\sqrt{1-m^2})\right] dx' = \frac{x}{m}\left[K - E\right] \tag{B7}$$

The terms resulting from the integration of $\cosh^{-1}\left|\frac{1}{\zeta}\right|$ will be removed by a superposition of other solutions according to equation (18) to avoid an infinite angle of attack at the center line. Such terms are denoted by the symbol (∞) in the expression for $\frac{z_1}{x}$.

The only term of w_1 not yet considered is $\frac{1}{m}\sqrt{1-\zeta^2}$. Inserting this term in (B6) yields

$$-|\zeta|\int_1^{|\zeta|} \frac{1}{m}\frac{\sqrt{1-\zeta'^2}}{\zeta^2}\, d\zeta' = \frac{1}{m}\left[\sqrt{1-\zeta^2} - |\zeta| \cos^{-1}\zeta\right] \tag{B8}$$

Thus from (B6), (B7), and (B8)

$$\frac{z_1}{x} = \frac{1}{m}\left[\sqrt{1-\zeta^2} - |\zeta| \cos^{-1}\zeta\right] + \frac{1}{m}\left[K\left(\sqrt{1-m^2}\right) - E\left(\sqrt{1-m^2}\right)\right]$$

$$-\frac{1}{m}(\infty)$$

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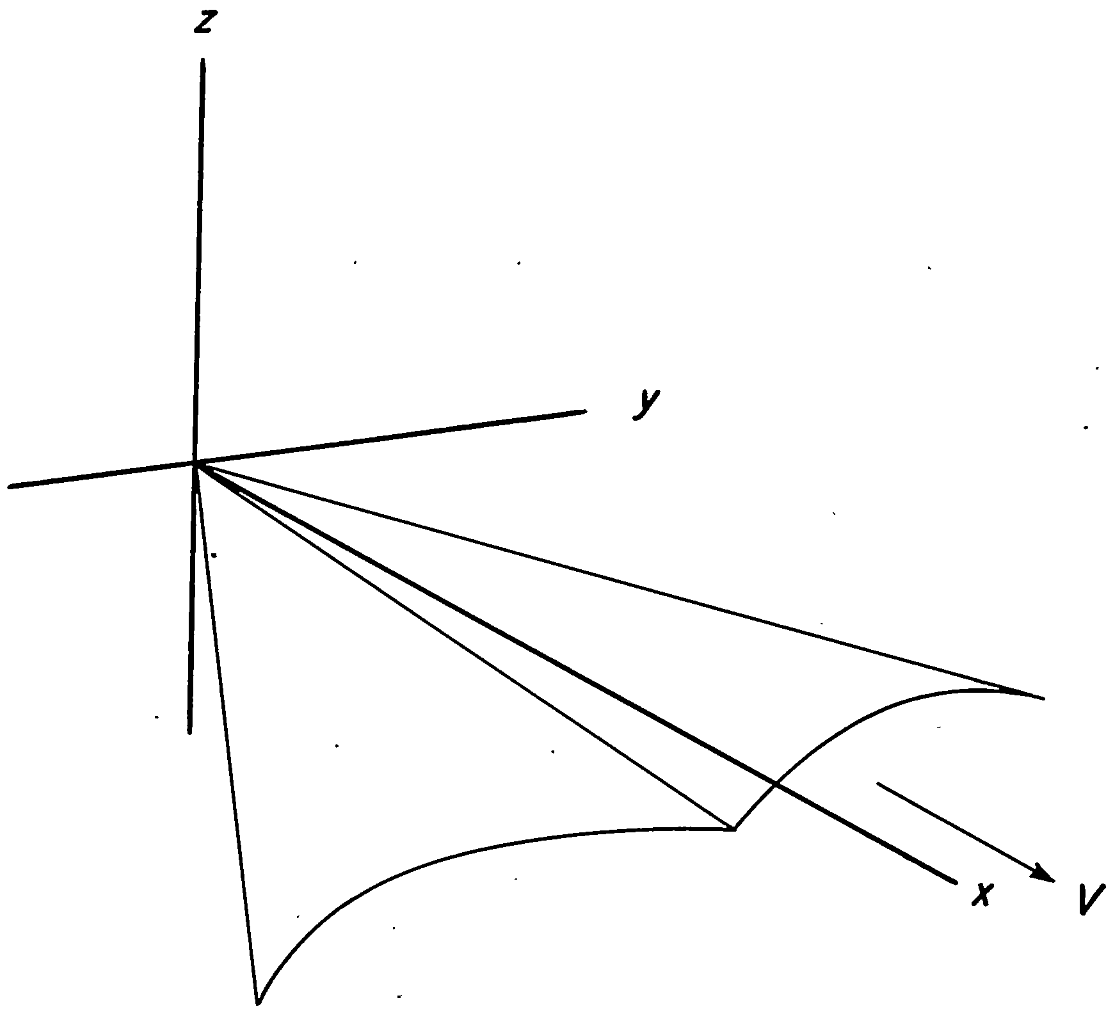

Figure 1.- Cambered triangular wing at an angle of attack showing the coordinate system.

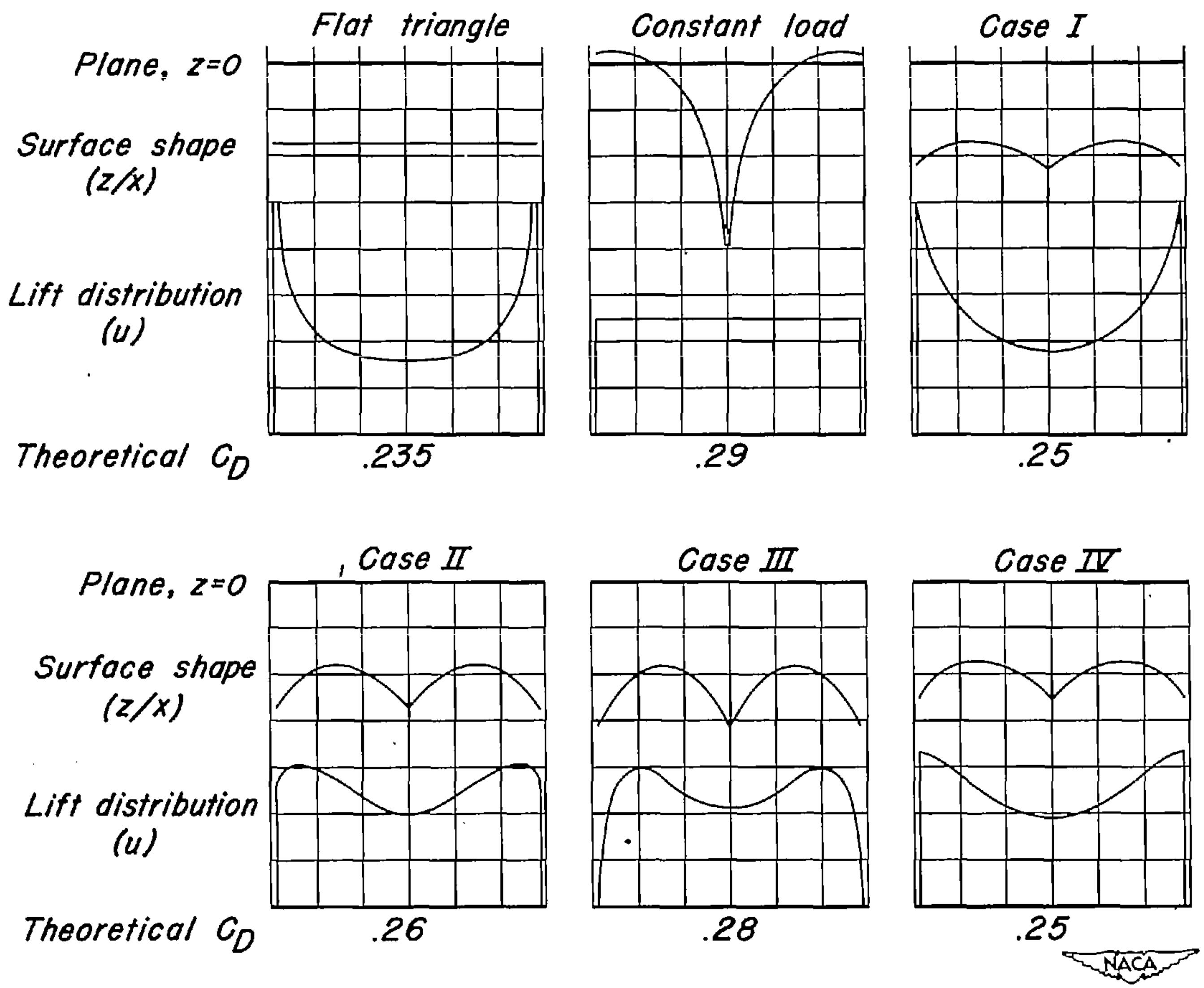

Figure 2.– Cross sections of wing surface shapes and lift distributions (C_L=1).